Information, Commitment, and War
Online Appendix: Proofs*

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Post-Shift Histories

We begin by characterizing behavior in post-shift continuations, which follows one of four histories, h_1-h_4. These histories describe the state of A's information about B's type, the number of previous battles, and whether a shift in power has occurred. In h_1, there has been one previous battle and A knows B's type; in h_2, there has been one previous battle and A does not know B's type; in h_3, there have been no previous battles and A knows B's type; and in h_4, there have been no previous battles and A does not know B's type. Histories place no restrictions on the timing or ordering of previous battles and belief updating. We focus on stationary Perfect Bayesian strategies in which players make the same choices whenever they have identical beliefs and the rest of the game is structurally identical. Finally, recall our restriction over the size of per-battle costs to ensure that there are no corner proposals, e.g. $x_n = 1$, in post-shift histories. We do this to highlight the fact that the shift in power *can* create corner proposals in pre-shift histories, as discussed below.

History h_1

At history h_1, A knows B's type, so its beliefs are either $\beta = 0$ or $\beta = 1$. Attack or rejection ends the game in a decisive battle. If B accepts a proposal x_{1T}, then stationarity implies that A can

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propose x_{1T} in all future periods, and B will accept it. A player of type b_T accepts some x_{1T} rather than reject iff

$$\frac{1-x_{1T}}{1-\delta} \geq \left(1-\frac{p}{1+s}\right)\frac{1}{1-\delta} - b_T \Leftrightarrow x_{1T} \leq \frac{p}{1+s} + (1-\delta)b_T. \tag{1}$$

Thus, A's most preferred proposal that is also acceptable to type b_T is $x^*_{1T} = p/(1+s)+(1-\delta)b_T$, because it is the minimum required to win acceptance. Proposing more provokes rejection, while proposing less wins acceptance of a less attractive proposal. (By this logic, A always proposes to keep as much as possible when it sets equilibrium proposals, so we only mention it in detail here.) A proposes x^*_{1T} rather than attack iff

$$\frac{x^*_{1T}}{1-\delta} \geq \frac{p}{1+s} \times \frac{1}{1-\delta} - a \Leftrightarrow a + b_T > 0,$$

which is true by assumption. Therefore, A's equilibrium proposal at history h_1 is $x^*_{1T} = p/(1+s)+(1-\delta)b_T$, which B accepts.

History h_2

At history h_2, A has not yet updated its beliefs, so $\beta = q$. As with h_1, attack or rejection ends the game in a decisive battle. This ensures that player-type b_T's payoffs to rejection are the same as in history h_1. A's available actions are to attack, which ends the game; to make a pooling proposal, which both types accept and keeps the game in history h_2; and to make a separating proposal, which the weak accepts and the strong rejects, and which moves the game to history h_1 upon acceptance.

By the logic of history h_1, A's optimal pooling proposal will be $y^*_{2L} = p/(1+s)+(1-\delta)b_L$. Since the strong type accepts x^*_{2L}, so must the weak type, since its payoff to rejection is lower than the strong type's, i.e. $b_H > b_L$. Further, Inequality (1), also ensures that A prefers x^*_{2L} to attack. Thus, forcing a battle is never sequentially rational at history h_2.

For a separating proposal, the weak type accepts and the strong rejects, and the incentive compatibility constraints are

$$1-x_{2H}+\delta\frac{1-x_{1H}^*}{1-\delta} \geq \left(1-\frac{p}{1+s}\right)\frac{1}{1-\delta}-b_H \qquad (b_H)$$
$$\left(1-\frac{p}{1+s}\right)\frac{1}{1-\delta}-b_L > 1-x_{2H}+\delta\frac{1-x_{1H}^*}{1-\delta} \qquad (b_L),$$

which reduce to

$$1+\delta\frac{1-x_{1H}^*}{1-\delta}-\left(1-\frac{p}{1+s}\right)\frac{1}{1-\delta}+b_L < x_{2H} \leq 1+\delta\frac{1-x_{1H}^*}{1-\delta}-\left(1-\frac{p}{1+s}\right)\frac{1}{1-\delta}+b_H.$$

A prefers to make the largest possible proposal satisfying these constraints, so its optimal separating proposal, after substitution, is $x_{2H}^* = p/(1+s)+(1-\delta)b_H$, which is the same as the proposal it makes to satisfy the weak type in history h_1, or x_{1H}^*. Again, the logic of h_1 implies that A prefers proposing x_{2H}^* to attacking.

We now derive the conditions under which A prefers to make each proposal. A prefers making a pooling proposal in each period over a deviation to the separating proposal iff

$$\frac{x_{1L}^*}{1-\delta} \geq \beta\left(\frac{p}{1+s}\times\frac{1}{1-\delta}-a\right)+(1-\beta)\frac{x_{1H}^*}{1-\delta} \Leftrightarrow \beta \geq \frac{b_H-b_L}{a+b_H} \equiv \overline{\beta},$$

and it makes a separating proposal when $\beta < \overline{\beta}$. Conversely, A prefers a separating proposal to a one-period deviation to a pooling proposal when $U_A(x_{2H}^*) > x_{2L}^* + \delta U_A(x_{2H}^*)$, or when $\beta < (b_H - b_L)/(a+b_H) = \overline{\beta}$. Therefore, at history h_2, A proposes x_{2H}^* (accepted by b_H and rejected by b_L) when $\beta < \overline{\beta}$ and x_{2L}^* (accepted by b_T) when $\beta \geq \overline{\beta}$.

History h_3

At history h_3, no battles have occurred, and A knows B's type, so its beliefs are $\beta = 0$ or $\beta = 1$. Acceptance of a proposal keeps the game in history h_3, while rejection or attack transitions the game to history h_1. If B accepts a proposal x_{3T}, then stationarity implies that A can propose x_{3T} in all future periods, and B will accept it. However, since rejection does not end the game at histories with no previous battles, B's rejection of any proposal that it should accept is out of equilibrium, and we stipulate that A believes $\beta' = 1$ in the event of any unexpected rejection. In other words, by rejecting a proposal, player-type b_T convinces A that it faces the strong type, and A behaves acordingly at history h_1. Therefore, player-type b_T will accept some x_{3T} iff

$$\frac{1-x_{3T}}{1-\delta} \geq -b_T + \delta\frac{1-x^*_{1L}}{1-\delta} \Leftrightarrow x_{3T} \leq 1+(1-\delta)b_T - \delta(1-x^*_{1L}).$$

A's most preferred proposal that is acceptable to player-type b_T is $x^*_{3T} = 1+(1-\delta)b_T - \delta(1-x^*_{1L})$, and A makes such a proposal in each period rather than attack iff

$$\frac{x^*_{3T}}{1-\delta} \geq -a + \delta\frac{x^*_{1T}}{1-\delta} \Leftrightarrow a + b_T \geq 0,$$

which is true by assumption. Likewise, A strictly prefers a one-period deviation to x^*_{3T} over attacking in every period iff

$$x^*_{3T} + \delta(-a) + \delta^2\left(\frac{p}{1+s} \times \frac{1}{1-\delta} - a\right) \geq -a + \delta\left(\frac{p}{1+s} \times \frac{1}{1-\delta} - a\right) \Leftrightarrow a + b_T \geq 0,$$

which is true by assumption. Therefore, A's equilibrium proposal at history h_3 is $x^*_{3T} = 1+(1-\delta)b_T - \delta(1-x^*_{1L})$, which type b_T accepts.

History h_4

At history h_4, no battles have occurred, and A is uncertain over which type it faces, so $\beta = q$. If A attacks, the game transitions to history h_2. If A makes a separating proposal, the game transitions to history h_1 upon rejection and history h_3 upon acceptance. If it makes a pooling proposal, the game remains in history h_4. Recall that A responds to any out-of-equilibrium rejection by believing $\beta' = 1$, which transitions the game to history h_1.

A's optimal pooling proposal will be the largest proposal that ensures the strong type's acceptance, and this will also ensure the weak type's acceptance. To verify this, note that the strong type accepts a proposal x_{4L} in every period iff

$$\frac{1-x_{4L}}{1-\delta} \geq -b_L + \delta\frac{1-x^*_{1L}}{1-\delta} \Leftrightarrow x_{4L} \leq 1+(1-\delta)b_L - \delta(1-x^*_{1L}).$$

The weak type, on the other hand, receives $-b_H + \delta(1-x^*_{1L})/(1-\delta)$ from rejection, since its out-of-equilibrium rejection convinces A that $\beta' = 1$, which is strictly less than the strong type's payoff to rejection. Therefore, player-type b_H accepts all proposals acceptable to b_L, and A sets its pooling proposal at $x^*_{4L} = 1 + (1-\delta)b_L - \delta(1 - x^*_{1L})$, which both types of B accept.

A separating proposal is accepted by the weak type but rejected by the strong. The weak type must prefer not to mimic the strong by rejecting and receiving more generous deals in the future, while the strong type must prefer to reject rather than mimic the weak type. To define the strong type's payoff for accepting, note that should player-type b_L deviate from the strategy of rejection by accepting a separating proposal, A believes $\beta = 0$ and proposes x^*_{3H} in the next period, which player-type b_L, returning to optimal play, rejects. This engages A's out-of-equilibrium beliefs $\beta' = 1$ for the next period, in which it proposes x^*_{1L}, which by stationarity B accepts in every subsequent period. Therefore, the incentive compatibility constraints are

$$\begin{aligned} 1 - x_{4H} + \delta\frac{1-x^*_{3H}}{1-\delta} &\geq -b_H + \delta\frac{1-x^*_{1L}}{1-\delta} && (b_H) \\ -b_L + \delta\frac{1-x^*_{1L}}{1-\delta} &> 1 - x_{4H} + \delta(-b_L) + \delta^2\frac{1-x^*_{1L}}{1-\delta} && (b_L), \end{aligned}$$

which reduce to

$$1 - \delta(1 - x^*_{1L}) + (1-\delta)b_L < x_{4H} \leq 1 - \frac{\delta}{1-\delta}(x^*_{3H} - x^*_{1L}) + b_H.$$

This inequality holds as long as $b_H > b_L$ and $0 < \delta < 1$, so A's optimal separating proposal is $x^*_{4H} = 1 - \delta(x^*_{3H} - x^*_{1L})/(1-\delta) + b_H$, which the strong type rejects and the weak accepts.

Now consider A's strategy. Its payoff to attacking at history h_4, which transitions the game to history h_2, depends on the value of β. Recall that, at history h_2, A makes a separating proposal when $\beta < \overline{\beta}$ and a pooling proposal when $\beta \geq \overline{\beta}$. Our first step is to show that, for all values of β, A will not attack. This allows for a subsequent comparison between separating and pooling proposals. Begin with $\beta < \overline{\beta}$, where A's payoff to attack is

$$U_A(\text{att}_4 | \beta < \overline{\beta}) = -a + \delta\left[\beta\left(\frac{p}{1+s} \times \frac{1}{1-\delta} - a\right) + (1-\beta)\left(x^*_{2H} + \delta\frac{x^*_{1H}}{1-\delta}\right)\right],$$

which is less attractive than a separating proposal, x^*_{4H}, iff

$$\beta\left(-a + \delta\frac{x^*_{1L}}{1-\delta}\right) + (1-\beta)\left(x^*_{4H} + \delta\frac{x^*_{3H}}{1-\delta}\right) > U_A(\text{att}_4 | \beta < \overline{\beta}) \Leftrightarrow \beta < \frac{b_H - b_L}{1 + b_H} = \overline{\beta}.$$

Therefore, when $\beta < \overline{\beta}$, making a separating proposal dominates attack. Next, when $\beta \geq \overline{\beta}$, A's payoff to attack is $U_A(\text{att}_4|\beta \geq \overline{\beta}) = -a + \delta x^*_{2L}/(1-\delta)$, which is less attractive than a separating proposal iff

$$\beta\left(-a+\delta\frac{x^*_{1L}}{1-\delta}\right)+(1-\beta)\left(x^*_{4H}+\delta\frac{x^*_{3H}}{1-\delta}\right) > -a+\delta\frac{x^*_{2L}}{1-\delta},$$

or when $\beta < 1$, $b_H > b_L > 0$, and $a > 0$. Therefore, A strictly prefers separating proposals to attacking at history h_4 for all values of β.

Finally, A prefers the pooling proposal in each period to the separating proposal iff

$$\frac{x^*_{4L}}{1-\delta} \geq \beta\left(-a+\delta\frac{x^*_{1L}}{1-\delta}\right)+(1-\beta)\left(x^*_{4H}+\delta\frac{x^*_{3H}}{1-\delta}\right) \Leftrightarrow \beta \geq \frac{b_H-b_L}{1+a+b_H} \equiv \underline{\beta},$$

and prefers the separating proposal iff $\beta < \underline{\beta}$. Likewise, A prefers making the separating proposal to a one-period deviation to the pooling proposal iff $U_A(x^*_{4H}) > x^*_{4L} + \delta U_A(x^*_{4H})$, or when $\beta < (b_H - b_L)/(1+a+b_H) = \underline{\beta}$. Therefore, in each period where the history is h_4, A proposes x^*_{4H} when $\beta < \underline{\beta}$ and x^*_{4L} when $\beta \geq \underline{\beta}$, type b_L accepts $x_{4L} \leq x^*_{4L}$, and type b_H accepts $x_{4H} \leq x^*_{4H}$.

Before considering pre-shift histories, it is worth noting that $\underline{\beta} < \overline{\beta}$, or that A is generally more willing to make separating proposals after one battle (history h_2) than after no battles (h_4). The discussion of equilibrium strategies given the four possible post-shift histories implies that, first, when A faces only one type of B after a shift in power, it makes an acceptable proposal to that type of B in all subsequent periods. Second, when A remains uncertain over which type it faces, there are three cases to consider: (a) when $\beta \geq \overline{\beta}$, A makes a pooling proposal that both types accept in each period; (b) when $\underline{\beta} \leq \beta < \overline{\beta}$, A makes pooling proposals in each period if there have been no battles but makes a separating proposal if there has been one battle; and (c) when $\beta < \underline{\beta}$, A makes separating proposals.

Pre-Shift Histories

We now move back up the game tree to consider pre-shift histories, h_5-h_7, and we characterize equilibrium behavior in the following format. For each history, we begin with a definition of equilibrium proposals that identify, first, the pooling proposal acceptable to both types of B and, second, the separating proposal acceptable to the weak type but unacceptable to the strong. Then, for each history we identify the conditions under which A will choose each of the options available to it—the pooling proposal, the separating proposal, and attack. Note that in pre-shift play, attack can be sequentially rational. Because A's share of the benefits in any given round cannot exceed one, for some values of s, A cannot receive enough of the benefits in the current period to prevent it from acting to forestall the shift. For each of the subsequent histories we identify when this constraint on x binds as a function of s, i.e. when the shift in power will be so large that there exists no proposal that can induce A not to forestall the shift by attacking. Given anticipated sequentially rational behavior and beliefs established above, this will complete the characterization of PBE for each part of the parameter space. We begin with histories at which one battle has been fought, h_5 and h_6, then consider history h_7, at which A's first move occurs.

History h_5

At history h_5, one battle has been fought, and A knows B's type. Rejection or attack ends the game in a decisive battle and prevents a shift in power from occurring, while acceptance leads to a shift in power and history h_1, where the next battle is decisive. B accepts pre-shift proposals x_{5T} that satisfy

$$1 - x_{5T} + \delta \frac{1 - x^*_{1T}}{1-\delta} \geq \frac{1-p}{1-\delta} - b_T \Leftrightarrow x_{5T} \leq 1 - \frac{1}{1-\delta}(1 - p - \delta(1 - x^*_{1T})) + b_T. \qquad (2)$$

A's possible replies are to attack or propose x^*_{5T}, allowing a shift in power and engaging strategies that follow history h_1. However, B cannot commit not to use its increased bargaining power after a shift, and as such it has incentives to be very generous to A before a shift in power (especially as $\delta \to 1$). The most it can promise to A in the present is $x_{5T} = 1$, and A chooses to attack rather than make such a proposal when

$$\frac{p}{1-\delta} - a > 1 + \delta \frac{x^*_{1T}}{1-\delta} \Leftrightarrow s > \frac{p\delta}{p+\delta-(1-\delta)(a+\delta b_T)-1} - 1. \tag{3}$$

Otherwise, there is sure to exist some proposal satisfying Inequality (2) that A prefers to make rather than attack. To see this, note that if A proposes some $x^\dagger_{5T}$ that satisfies (2) at equality, then $x^\dagger_{5T} + \delta x^*_{1T}/(1-\delta) \geq p/(1-\delta) - a$ is strictly true, but it may not the the case that $x^\dagger_{5T} \leq 1$. But when s is small enough that (3) does not hold, A's equilibrium proposal x^*_{5T} is therefore the minimum of the constraint in Inequality (2) and 1. Formally, its equilibrium proposal is

$$x^*_{5T} = \min\{1 - \frac{1}{1-\delta}(1-p-\delta(1-x^*_{1T})) + b_T, 1\}.$$

More generally for all pre-shift proposals, s can be so large that equilibrium proposals can have corner solutions, i.e. $x_n = 1$, but rather than re-derive it for each proposal, we simply note in the remainder of the proof that A will either meet the proper constraint at equality or set it at 1 when s is sufficiently large.

Finally, Inequality (3) shows that A's choice over attack or making an acceptable proposal depends on the player-type it believes that it faces at this information set, b_T. When $\beta = 1$, A attacks iff

$$s > \frac{p\delta}{p+\delta-(1-\delta)(a+\delta b_L)-1} - 1 \equiv \underline{s} \tag{4}$$

and otherwise proposes x^*_{5L}, which the strong type accepts. When $\beta = 0$, A attacks iff

$$s > \frac{p\delta}{p+\delta-(1-\delta)(a+\delta b_H)-1} - 1 \equiv \hat{s} \tag{5}$$

and otherwise proposes x^*_{5H}, which the weak type accepts.

History h_6

At history h_6, there has been one battle, and A does not know which type of B it faces. Rejection or attack again ends the game in a decisive battle, and acceptance engages either history h_1 (A

knows B's type) or h_2 (A does not know B's type), depending on whether A makes a separating or pooling proposal. Should B accept a proposal, the consequences depend on its type and on the response of the other type to the same proposal (i.e. separating or pooling).

To define the pooling proposal, we begin with the strong type's acceptance rule, which is conditional on the value of β, since the weak will also accept and engage history h_2. However, regardless of whether A makes a separating or pooling proposal after history h_2, player-type b_L receives an equivalent payoff—a decisive battle in response to a separating proposal or its certainty equivalent, x^*_{2L}, in a pooling proposal—and therefore we can define a single proposal required to win the strong type's acceptance for any value of β. Using x^*_{2L} to simplify the presentation, the strong type accepts proposals satisfying

$$1 - x_{6L} + \delta \frac{1 - x^*_{2L}}{1-\delta} \geq (1-p)\frac{1}{1-\delta} - b_L \Leftrightarrow x_{6L} \leq 1 - \frac{1}{1-\delta}(1 - p - \delta(1 - x^*_{2L})) + b_L.$$

The weak type accepts such a proposal when

$$1 - x_{6L} + \delta \frac{1 - x^*_{2L}}{1-\delta} \geq (1-p)\frac{1}{1-\delta} - b_H,$$

where its payoff differs from the strong type's in that it pays greater per-battle costs, $b_H > b_L$. Therefore, since player-type b_H is worse off by rejecting, it is sure to accept all proposals also acceptable to the strong type. To make a pooling proposal, then, A sets

$$x^*_{6L} = \min\{1 - \frac{1}{1-\delta}(1 - p - \delta(1 - x^*_{2L})) + b_L, 1\}.$$

Separating proposals induce the weak type's acceptance and the strong type's rejection. Acceptance transitions the game into history h_1, at which A believes $\beta = 0$ and proposes x^*_{1H}, which b_H accepts, in all post-shift periods. The incentive compatibility constraints are

$$1 - x_{6H} + \delta \frac{1 - x^*_{1H}}{1-\delta} \geq p \times \frac{1}{1-\delta} - b_H \qquad (b_H)$$

$$p \times \frac{1}{1-\delta} - b_L > 1 - x_{6H} + \delta\left(\frac{p}{1+s} \times \frac{1}{1-\delta} - b_L\right) \qquad (b_L),$$

which reduce to

$$1 + \delta\left(\frac{p}{1+s} \times \frac{1}{1-\delta} - b_L\right) - \frac{p}{1-\delta} + b_L < x_{6H} \leq 1 - \frac{1}{1-\delta}(1 - p - \delta(1 - x^*_{1H})) + b_H \qquad (6)$$

Inequality (6) holds as long as $b_H > b_L$, ensuring that A can set proposals to induce rejection by the strong and acceptance by the weak. Therefore, to make a separating proposal, A sets

$$x^*_{6H} = \min\{1 - \frac{1}{1-\delta}(1-p-\delta(1-x^*_{1H})) + b_H, 1\}.$$

A's has three choices: attack, a separating proposal, and a pooling proposal. While A's payoffs to attacking or making a separating proposal do not depend on β, its payoff to making a pooling proposal does, since doing so moves the game into history h_2. Thus, we first characterize A's best reply when $\beta \geq \overline{\beta}$ and follow by discussing $\beta < \overline{\beta}$.

When $\beta \geq \overline{\beta}$, A's payoff from making a separating proposal is $U_A(x^*_{6H})$, or

$$U_A(x^*_{6H}) = \beta\left(\frac{p}{1-\delta} - a\right) + (1-\beta)\left(x^*_{6H} + \delta\frac{x^*_{1H}}{1-\delta}\right),$$

where its post-shift per-period payoff of x^*_{1H} is derived from history h_1. A's payoff for a pooling proposal is $U_A(x^*_{6L}) = x^*_{6L} + \delta x^*_{2L}/(1-\delta)$, where its post-shift per-period payoff of x^*_{2L} is derived from history h_2. However, A prefers attacking to the most generous possible pooling proposal, $x_{6L} = 1$, today iff

$$\frac{p}{1-\delta} - a > 1 + \delta\frac{x^*_{2L}}{1-\delta} \Leftrightarrow s > \underline{s},$$

where $\underline{s}$ is the same constraint as $\underline{s}$ defined above. Further, A prefers attack to the constrained maximum separating proposal, $x_{6H} = 1$, iff

$$\frac{p}{1-\delta} - a > \beta\left(\frac{p}{1-\delta} - a\right) + (1-\beta)\left(1 + \delta\frac{x^*_{1H}}{1-\delta}\right) \Leftrightarrow s > \hat{s},$$

as defined above. Note that, since $b_L < b_H$, it is also the case that $\hat{s} > \underline{s}$. Finally, when A does not prefer war to the most generous pooling proposal, or $s \leq \underline{s}$, we compare the separating and pooling proposals, and algebra shows that A never prefers the optimal separating proposal, x^*_{6H} as long as $\beta > (b_H - b_L)/(a + b_H) = \overline{\beta}$, which is the constraint that defines this subcase. Therefore, at history h_6 when $\beta \geq \overline{\beta}$, A's best responses are summarized as follows. A proposes x^*_{6L} when $s \leq \underline{s}$, x^*_{6H} when $\underline{s} < s \leq \hat{s}$ and attacks when $s > \hat{s}$.

When $\beta < \overline{\beta}$, A's payoffs for attack and separating proposals remain unchanged, and its payoff for making a pooling proposal and entering history h_2 is $U_A(x^*_{6L})$, or

$$U_A(x^*_{6L}) = x^*_{6L} + \delta\left[\beta\left(\frac{p}{1+s} \times \frac{1}{1-\delta} - a\right) + (1-\beta)\left(x^*_{2H} + \delta\frac{x^*_{1H}}{1-\delta}\right)\right].$$

However, no such proposal—not even $x_{6L} = 1$ can induce A not to attack iff

$$\frac{p}{1-\delta} - a > 1 + \delta\left[\beta\left(\frac{p}{1+s} \times \frac{1}{1-\delta} - a\right) + (1-\beta)\left(x^*_{2H} + \delta\frac{x^*_{1H}}{1-\delta}\right)\right],$$

or when

$$s > -\frac{(-1+\delta)\left(-1+p+a(-1+\beta\delta)+(-1+\beta)\delta b_H\right)}{1-p-\delta+a(-1+\delta)(-1+\beta\delta)+(-1+\beta)(-1+\delta)\delta b_H}. \tag{7}$$

Algebra shows that the constraint in Inequality (7) is less than $\hat{s}$, and the pooling proposal exists only when (7) does not hold. As in the previous subcase, there exists a separating proposal only when $s < \hat{s}$, and A prefers attack when $s > \hat{s}$. However, when A does not prefer war to the most generous pooling proposal, i.e. when (7) does not hold, it strictly prefers to make the separating proposal iff $U_A(x^*_{6H}) > U_A(x^*_{6L})$, or $\beta < (b_H - b_L)/(a + b_H) = \overline{\beta}$, which is the constraint that defines the case. When $\underline{s} \le s < \hat{s}$ A prefers attack to the pooling proposal but not to the separating. Therefore, at history h_6 when $\beta < \overline{\beta}$, A attacks when $s > \hat{s}$ and makes the separating proposal x^*_{6H} when $s \le \hat{s}$.

To summarize results thus far, taking into account A's equilibrium behavior from histories h_5 and h_6, the equilibrium space is two-dimensional. First, when $\beta = 1$ (h_5), A attacks when $s > \underline{s}$ and proposes x^*_{5L} in order to transition to h_1 when $s \le \underline{s}$. Second, when $\beta = 0$ (h_5), A attacks when $s > \hat{s}$ and proposes x^*_{5H} in order to transition to h_1 when $s \le \hat{s}$. Third, when $\beta \ge \overline{\beta}$ at history h_6, A proposes x^*_{6L} in order to transition to history h_2 when $s \le \underline{s}$, proposes x^*_{6H} in order to transition to history h_1 when $\underline{s} < s \le \hat{s}$; and attacks when $s > \hat{s}$. Fourth, when $\beta < \overline{\beta}$ at history h_6, A attacks when $s > \hat{s}$ and proposes x^*_{6H} in order to transition into history h_1 upon acceptance when $s \le \hat{s}$. Finally, it is worth noting that, for b_H, b_L, and a sufficiently low, it is also the case that $\hat{s} > \underline{s}$. We maintain these restrictions through the remainder of the proofs.

History h_7

We now consider history h_7, which corresponds to A's initial decision node and B's first response to a proposal. At this history no battles have been fought, and A is uncertain over which type it faces, beginning the game with prior beliefs $\beta = q$. The analysis of histories h_1-h_6 imply that, with A's initial choices of attacking, making a separating proposal, and making a pooling proposal, there are nine combinations of parameter values to consider that imply unique sets of choices and outcomes down the path of play. Below, when we say that A will "satisfy the strongest type present," we mean that it issues a pooling proposal if it faces both types and a proposal acceptable to whatever type is present if it faces only one. The nine cases are:

1. $\beta \geq \overline{\beta}$ and $s \leq \underline{s}$. *Post-battle, pre-shift*: satisfy the strongest type present. *Post-shift*: satisfy the strongest type present.

2. $\beta \geq \overline{\beta}$ and $\underline{s} < s \leq \hat{s}$. *Post-battle, pre-shift*: if facing both types, make a separating proposal; if one type, attack the strong but satisfy the weak. *Post-shift*: satisfy the strongest type present.

3. $\beta \geq \overline{\beta}$ and $s > \hat{s}$. *Post-battle, re-shift*: attack regardless of beliefs. *Post-shift*: satisfy the strongest type present.

4. $\underline{\beta} \leq \beta < \overline{\beta}$ and $s \leq \underline{s}$. *Post-battle, pre-shift*: if facing both types, make a separating proposal; if one type, satisfy it. *Post-shift*: if both types, make a separating after one battle but a pooling proposal after no battles; if one type, satisfy it.

5. $\underline{\beta} \leq \beta < \overline{\beta}$ and $\underline{s} < s \leq \hat{s}$. *Post-battle, pre-shift*: if both types, make a separating proposal; if one type, attack the strong but satisfy the weak. *Post-shift*: if both types, make a separating after one battle but a pooling proposal after no battles; if one type, satisfy it.

6. $\underline{\beta} \leq \beta < \overline{\beta}$ and $s > \hat{s}$. *Post-battle, pre-shift*: attack regardless of beliefs. *Post-shift*: if both types, make a separating after one battle but a pooling proposal after no battles; if one

type, satisfy it.

7. $\beta < \underline{\beta}$ and $s \leq \underline{s}$. *Post-battle, pre-shift*: if both types, make a separating proposal; if one type, satisfy it. *Post-shift*: if both types, make a separating proposal; if one type, satisfy it.

8. $\beta < \underline{\beta}$ and $\underline{s} < s \leq \hat{s}$. *Post-battle, pre-shift*: if both types, make a separating proposal; if one type, attack the strong but satisfy the weak. *Post-shift*: if both types, make a separating proposal; if one type, satisfy it.

9. $\beta < \underline{\beta}$ and $s > \hat{s}$. *Post-battle, pre-shift*: attack regardless of beliefs. *Post-shift*: if both types, make a separating proposal; if one type, satisfy it.

As above, we specify each player-type's best response rules as a function of anticipated strategies and beliefs (which we showed to be consistent and sequentially rational above), which are conditional on the parameter values of β and s that define each of the nine cases. We begin with each type of B's acceptance constraints and then examine A's choice of attack, separating proposal, and pooling proposal in turn, following the format established in histories h_5 and h_6.

Case 1. $\beta \geq \overline{\beta}$ **and** $s \leq \underline{s}$**.**

Post-battle, pre-shift: satisfy the strongest type present. *Post-shift*: satisfy the strongest type present.

First, define a pooling proposal as one that both types of B accept, which implies that any type's rejection is out-of-equilibrium. This engages A's out-of-equilibrium beliefs, $\beta' = 1$. The strong type accepts when

$$1 - x_{7L} + \delta \frac{1 - x_{4L}^*}{1-\delta} \geq -b_L + \delta(1 - x_{5L}^*) + \delta^2 \frac{1 - x_{1L}^*}{1-\delta},$$

or when

$$x_{7L} \leq 1 - \delta\left((1 - x_{5L}^*) + \delta \frac{1 - x_{1L}^*}{1-\delta} - \frac{1 - x_{4L}^*}{1-\delta}\right) + b_L. \tag{8}$$

The weak type accepts such a proposal when

$$1 - x_{7L} + \delta \frac{1 - x_{4L}^*}{1-\delta} \geq -b_H + \delta(1 - x_{5L}^*) + \delta^2 \frac{1 - x_{1L}^*}{1-\delta},$$

where its payoffs differ only from the strong type's in that it must may a higher per-battle cost if it rejects a proposal, $b_H > b_L$, to go on to accept proposals consistent with A's out-of-equilibrium beliefs $\beta' = 1$. Therefore, since the weak is worse off by rejecting, it is sure to accept all proposals also acceptable to the strong type, and to make a pooling proposal that both types accept, A sets

$$x_{7L}^* = \min\{1 - \delta \left((1 - x_{5L}^*) + \delta \frac{1 - x_{1L}^*}{1-\delta} - \frac{1 - x_{4L}^*}{1-\delta} \right) + b_L, 1\}.$$

Next, define a separating proposal as one that only the weak type of B accepts, leading A to believe $\beta = 1$ upon rejection and $\beta = 0$ upon acceptance. First, the strong type must prefer to reject, honestly revealing its type, over accepting and mimicking the weak type; this results in the strong type rejecting A's subsequent proposals upon its return to optimal behavior, engaging A's out-of-equilibrium beliefs $\beta' = 1$ and finally guaranteeing itself a payoff based on x_{1L}^*. The incentive compatibility constraints are

$$\begin{aligned}
1 - x_{7H} + \delta \frac{1 - x_{3H}^*}{1-\delta} &\geq -b_H + \delta(1 - x_{5L}^*) + \delta^2 \frac{1 - x_{1L}^*}{1-\delta} && (b_H) \\
-b_L + \delta(1 - x_{5L}^*) + \delta^2 \frac{1 - x_{1L}^*}{1-\delta} &> 1 - x_{7H} + \delta(-b_L) + \delta^2 \frac{1 - x_{1L}^*}{1-\delta} && (b_L),
\end{aligned}$$

which reduce to

$$1 + (1-\delta) b_L - \delta(1 - x_{5L}^*) < x_{7H} \leq 1 - \delta \left((1 - x_{5L}^*) + \delta \frac{1 - x_{1L}^*}{1-\delta} - \frac{1 - x_{3H}^*}{1-\delta} \right) + b_H.$$

This inequality holds, ensuring the existence of a separating proposal, when $b_H > b_L$, which is true by construction. Therefore, A's optimal separating proposal is

$$x_{7H}^* = \min\{1 - \delta \left((1 - x_{5L}^*) + \delta \frac{1 - x_{1L}^*}{1-\delta} - \frac{1 - x_{3H}^*}{1-\delta} \right) + b_H, 1\}.$$

To characterize A's best reply, we follow the logic used above to identify when, as a function of s, there exist proposals that A prefers to attack. For this case, we first show that, when A does not prefer war to the most generous pooling proposal, it strictly prefers pooling proposals to separating proposals when $\beta \geq \overline{\beta}$, which is one of the constraints that defines the case. We then confirm that the pooling proposal exists in Case 1.

A prefers the pooling proposal when $U_A(x^*_{7L}) \geq U_A(x^*_{7H})$, or

$$x^*_{7L} + \delta \frac{x^*_{4L}}{1-\delta} \geq \beta \left(-a + \delta x^*_{5H} + \delta^2 \frac{x^*_{1L}}{1-\delta} \right) + (1-\beta) \left(x^*_{7H} + \delta \frac{x^*_{3H}}{1-\delta} \right) \Leftrightarrow \beta \geq \overline{\beta}.$$

Therefore, as long as the pooling proposal exists for all of Case 1, A will choose it in equilibrium, whether or not the separating proposal exists. To show when it exists, recall that the most that B can promise to accept in the first period is $x_{7L} = 1$, and when

$$1 + \delta \frac{x^*_{4L}}{1-\delta} \geq -a + \delta x^*_{6L} + \delta^2 \frac{x^*_{1L}}{1-\delta} \Leftrightarrow s \leq -1 + \frac{p\delta^2}{-1 + a(-1+\delta) + \delta(p+\delta) + (-1+\delta)\delta^2 b_L},$$

there is sure to be a proposal x^*_{7L}, as defined above, that A prefers to attacking. Finally, note that, as long as $0 < \delta < 1$, the above constraint is greater than $\underline{s}$, ensuring that, for Case 1, A's equilibrium strategy is to make a pooling proposal x^*_{7L}, which both types of B accept.

Case 2. $\beta \geq \overline{\beta}$ **and** $\underline{s} < s \leq \hat{s}$**.**

Post-battle, pre-shift: if facing both types, make a separating proposal; if one type, attack the strong but satisfy the weak. *Post-shift*: satisfy the strongest type present.

First, define a pooling proposal as one that both types of B accept, which implies that any type's rejection is out-of-equilibrium. This engages A's out-of-equilibrium beliefs, $\beta' = 1$. The strong type accepts iff

$$1 - x_{7L} + \delta \frac{1-x^*_{4L}}{1-\delta} \geq -b_L + \delta \left(\frac{1-p}{1-\delta} - b_L \right) \Leftrightarrow x_{7L} \leq 1 + \frac{\delta}{1-\delta}(p - x^*_{4L}) + (1+\delta) b_L. \tag{9}$$

The weak type accepts iff

$$1 - x_{7L} + \delta \frac{1-x^*_{4L}}{1-\delta} \geq -b_H + \delta \left(\frac{1-p}{1-\delta} - a \right),$$

where its payoffs differ from the strong type's only in terms of its per-battle costs, $b_H > b_L$. Therefore, since the weak is worse off by rejecting, it is sure to accept all proposals also acceptable to the strong type. To make a pooling proposal that both types accept, A sets

$$x^*_{7L} = \min\{1 + \frac{\delta}{1-\delta}(p - x^*_{4L}) + (1+\delta)b_L, 1\}.$$

Next, define a separating proposal as one that only the weak type of B accepts, leading A to believe $\beta = 1$ upon rejection and $\beta = 0$ upon acceptance. The incentive compatibility constraints are

$$1 - x_{7H} + \delta\frac{1-x^*_{3H}}{1-\delta} \geq -b_H + \delta\left(\frac{1-p}{1-\delta} - b_L\right) \qquad (b_H)$$

$$-b_L + \delta\left(\frac{1-p}{1-\delta} - b_L\right) > 1 - x + \delta(-b_L) + \delta^2\frac{1-x^*_{1L}}{1-\delta} \qquad (b_L),$$

which reduce to

$$1 + \delta^2\frac{1-x^*_{1L}}{1-\delta} + b_L - \delta\left(\frac{1-p}{1-\delta}\right) < x_{7H} \leq 1 + \frac{\delta}{1-\delta}(p - x^*_{3H}) + (1+\delta)b_H.$$

This inequality holds as long as $b_L < b_H$, which is true by assumption. Therefore, when it wishes to separate types, A sets

$$x^*_{7H} = \min\{1 + \frac{\delta}{1-\delta}(p - x^*_{3H}) + (1+\delta)b_H, 1\}.$$

To characterize A's best reply, we first show that, conditional on their existence, A prefers making a pooling over a separating proposal in Case 2 iff $U_A(x^*_{7L}) \geq U_A(x^*_{7H})$, or

$$x^*_{7L} + \delta\frac{x^*_{4L}}{1-\delta} \geq \beta\left(-a + \delta\left(\frac{p}{1-\delta} - a\right)\right) + (1-\delta)\left(x^*_{7H} + \delta\frac{x^*_{3H}}{1-\delta}\right) \Leftrightarrow \beta \geq (b_H - b_L)/(a + b_H) = \overline{\beta},$$

which is one of the constraints that defines Case 2.

We next show A only makes pooling proposals in Case 2, because it is guaranteed to exist in this case. By the logic used above, its preference for pooling over separating proposals means that it will always choose the former in equilibrium as long as s is not so large to preclude its existence. The most that B can promise to yield to A in the first period, $x_{7H} = 1$, and when

A prefers attack to such a generous proposal, then there exists no x_{7L} such that A cannot be induced not to attack. Therefore, a pooling proposal exists that it prefers to attack iff

$$1+\delta\frac{x_{4L}^*}{1-\delta} \geq -a+\delta\left[\beta\left(\frac{p}{1-\delta}-a\right)+(1-\beta)\left(x_{6H}^*+\delta\frac{x_{1H}^*}{1-\delta}\right)\right],$$

or

$$s \leq \frac{(1-\delta)\left(1+a+\delta-p\delta+a\beta\delta+(-1+\beta)\delta b_H+\delta(1+\delta)b_L\right)}{-1+\delta(p+\delta)+a(-1+\delta)(1+\beta\delta)+(-1+\delta)\delta\left((-1+\beta)b_H+(1+\delta)b_L\right)}.$$

The above constraint is greater than $\hat{s}$, which defines the case, as long as $0<\delta<1$. Therefore, in Case 2, A's best response is to make the pooling proposal x_{7L}^*, after which it does not update its beliefs.

Case 3. $\beta \geq \overline{\beta}$ **and** $s > \hat{s}$

Post-battle, re-shift: attack regardless of beliefs. *Post-shift*: satisfy the strongest type present.

Since Case 3 differs from Case 2 only in A's payoff for attack, equilibrium separating and pooling proposals are identical to those defined in Case 2. As a result, A will prefer to make the pooling proposal over the separating proposal when s is sufficiently low that they both exist. However, the ranges over which the proposals exist differ given A's new payoff for attack.

To determine A's best reply, we need only define the thresholds over s below which each proposal exists. As before, a pooling proposal exists that A prefers to attack iff

$$1+\delta x_{4L}^* 1-\delta \geq -a+\left(\frac{p}{1-\delta}-a\right) \Leftrightarrow s \leq -1+\frac{p\delta^2}{-1+\delta(p+\delta)+a\left(-1+\delta^2\right)+\delta\left(-1+\delta^2\right)b_L} = \tilde{s}.$$

Otherwise, when $s > \tilde{s}$, there exists no pooling proposal that A prefers to attack. A separating proposal exists iff

$$\beta\left[-a+\delta\left(\frac{p}{1-\delta}-a\right)\right]+(1-\beta)\left(1+\delta\frac{x_{3H}^*}{1-\delta}\right) \geq -a+\left(\frac{p}{1-\delta}-a\right),$$

or

$$s \leq -1+\frac{p\delta^2}{-1+\delta(p+\delta)+a\left(-1+\delta^2\right)+(-1+\delta)\delta\left(b_H+\delta b_L\right)} \equiv \overline{s}.$$

Otherwise, when $s > \overline{s}$, there exists no separating proposal that A prefers to attack.

Since $b_H > b_L$, it is also the case that $\overline{s} > \tilde{s}$. Therefore, we can characterize A's best response in Case 3 in the following way. When $\hat{s} \le s < \tilde{s}$, both proposals exist and A makes the pooling x^*_{7L}. When $\tilde{s} \le s < \overline{s}$, the separating proposal exists but the pooling does not, and since $s < \overline{s}$, A makes the separating x^*_{7H}. Finally, when $s > \overline{s}$, A attacks.

Case 4. $\underline{\beta} \le \beta < \overline{\beta}$ **and** $s \le \underline{s}$

Post-battle, pre-shift: if facing both types, make a separating proposal; if one type, satisfy it. *Post-shift*: if both types, make a separating after one battle but a pooling proposal after no battles; if one type, satisfy it.

First, define a pooling proposal as one that both types of B accept, which implies that any type's rejection is out of equilibrium. This engages A's out-of-equilibrium beliefs, $\beta' = 1$. The strong type accepts iff

$$1 - x_{7L} + \delta \frac{1 - x^*_{4L}}{1-\delta} \ge -b_L + \delta(1 - x^*_{5L}) + \delta^2 \frac{1 - x^*_{1L}}{1-\delta} \Leftrightarrow x_{7L} \le 1 + \delta \frac{1 - x^*_{4L}}{1-\delta} - \delta(1 - x^*_{5L}) - \delta^2 \frac{1 - x^*_{1L}}{1-\delta} + b_L.$$

The weak type accepts such a proposal iff

$$1 - x_{7L} + \delta \frac{1 - x^*_{4L}}{1-\delta} \ge -b_H + \delta(1 - x^*_{5L}) + \delta^2 \frac{1 - x^*_{1L}}{1-\delta},$$

and as before, the weak type's payoffs differ only in its lower payoff to rejection, which ensures that it accepts all proposals also acceptable to the strong type. To make a pooling proposal that both types accept, A sets

$$x^*_{7L} = \min\{1 + \delta \frac{1 - x^*_{4L}}{1-\delta} - \delta(1 - x^*_{5L}) - \delta^2 \frac{1 - x^*_{1L}}{1-\delta} + b_L, 1\}.$$

Next, define a separating proposal as one that only the weak type of B accepts, leading A to believe $\beta = 1$ upon rejection and $\beta = 0$ upon acceptance. Incentive compatibility requires that the weak accept and the strong reject, or

$$1 - x_{7H} + \delta\frac{1 - x^*_{3H}}{1-\delta} \geq -b_H + \delta(1 - x^*_{5L}) + \delta^2\frac{1 - x^*_{1L}}{1-\delta} \qquad (b_H)$$
$$-b_L + \delta(1 - x^*_{5L}) + \delta^2\frac{1 - x^*_{1L}}{1-\delta} > 1 - x_{7H} + \delta(-b_L) + \delta^2\frac{1 - x^*_{1L}}{1-\delta} \qquad (b_L),$$

which reduces to

$$1 - \delta(1 - x^*_{5L}) + (1-\delta)b_L < x_{7H} \leq 1 + \delta\frac{1 - x^*_{3H}}{1-\delta} - \delta(1 - x^*_{5L}) - \delta^2\frac{1 - x^*_{1L}}{1-\delta} + b_H.$$

This inequality holds, ensuring the existence of a separating proposal, when $b_H > b_L$, which is true by construction. Therefore, when it wishes to make a proposal that only the weak type accepts, A sets

$$x^*_{7H} = \min\{1 + \delta\frac{1 - x^*_{3H}}{1-\delta} - \delta(1 - x^*_{5L}) - \delta^2\frac{1 - x^*_{1L}}{1-\delta} + b_H, 1\}.$$

To characterize A's best reply, we first show that, conditional on their existence, A always prefers to make a pooling proposal over a separating one in Case 4. A prefers to propose x^*_{7L} iff $U_A(x^*_{7L}) \geq U_A(x^*_{7H})$, or

$$x^*_{7L} + \delta\frac{x^*_{4L}}{1-\delta} \geq \beta\left(-a + \delta x^*_{5L} + \delta^2\frac{x^*_{1L}}{1-\delta}\right) + (1-\beta)\left(x^*_{7H} + \delta\frac{x^*_{3H}}{1-\delta}\right) \Leftrightarrow \beta \geq \underline{\beta},$$

a constraint that helps define the case. Therefore, as long as the pooling proposal exists, A will make it in equilibrium in Case 4. The most that B can promise to yield to A in the current period is $x_{7L} = 1$, and iff $u_A(\text{att}_7) > u_A(x^*_{7L})$, there exists no nosneparating proposal that A prefers to attacking. A attacks iff

$$-a + \delta\left[\beta\left(\frac{p}{1-\delta} - a\right)\right] + (1-\beta)\left(x^*_{6H} + \delta\frac{x^*_{2H}}{1-\delta}\right) > 1 + \delta\frac{x^*_{4L}}{1-\delta},$$

or

$$s > \frac{(1-\delta)\left(1 + a + \delta - p\delta + a\beta\delta + (-1+\beta)\delta b_H + \delta(1+\delta)b_L\right)}{-1 + \delta(p+\delta) + a(-1+\delta)(1+\beta\delta) + (-1+\delta)\delta\left((-1+\beta)b_H + (1+\delta)b_L\right)}.$$

Algebra shows that, as long as $0 < \delta < 1$ (true by construction), the attack constraint falls above $\underline{s}$, which ensures that the attack constraint is never satisfied in Case 4, and A will make the pooling proposal regardless of whether the separating proposal exists. Therefore, in Case 4, A makes a pooling proposal x^*_{7L}, which both types of B accept.

Case 5. $\underline{\beta} \leq \beta < \overline{\beta}$ **and** $\underline{s} < s \leq \hat{s}$**.**

Post-battle, pre-shift: if both types, make a separating proposal; if one type, attack the strong but satisfy the weak. *Post-shift*: if both types, make a separating after one battle but a pooling proposal after no battles; if one type, satisfy it.

First, define a pooling proposal as one that both types of B accept, which implies that any type's rejection is out of equilibrium. This engages A's out-of-equilibrium beliefs, $\beta' = 1$. The strong type accepts when

$$1 - x_{7L} + \delta\frac{1-x^*_{4L}}{1-\delta} \geq -b_L + \delta\left(\frac{1-p}{1-\delta} - b_L\right) \Leftrightarrow x_{7L} \leq 1 + \delta\frac{1-x^*_{4L}}{1-\delta} - \delta\left(\frac{1-p}{1-\delta}\right) + (1+\delta)b_L.$$

The weak type also accepts when

$$1 - x_{7L} + \delta\frac{1-x^*_{4L}}{1-\delta} \geq -b_H + \delta\left(\frac{1-p}{1-\delta} - b_H\right).$$

As before, the weak type's payoffs differ only in its lower payoff to rejection, which ensures that it is sure to accept all proposals also acceptable to the strong type. To make a pooling proposal that both types accept, A sets

$$x^*_{7L} = \min\{1 + \delta\frac{1-x^*_{4L}}{1-\delta} - \delta\left(\frac{1-p}{1-\delta}\right) + (1+\delta)b_L, 1\}.$$

Next, define a separating proposal as one that only the weak type of B accepts, leading A to believe $\beta = 1$ upon rejection and $\beta = 0$ upon acceptance. The incentive compatibility constraints are

$$1 - x_{7H} + \delta\frac{1-x^*_{3H}}{1-\delta} \geq -b_H + \delta\left(\frac{1-p}{1-\delta} - b_H\right) \qquad (b_H)$$

$$-b_L + \delta\left(\frac{1-p}{1-\delta} - b_L\right) > 1 - x_{7H} + \delta(-b_L) + \delta^2\frac{1-x^*_{1L}}{1-\delta} \qquad (b_L),$$

which reduce to

$$1 + \delta^2\frac{1-x^*_{1L}}{1-\delta} + b_L - \delta\left(\frac{1-p}{1-\delta}\right) < x_{7H} \leq 1 + \delta\frac{1-x^*_{3H}}{1-\delta} - \delta\left(\frac{1-p}{1-\delta}\right) + (1+\delta)b_H.$$

This inequality holds, ensuring the existence of a separating proposal, when $b_L < b_H$, which is true by construction. Therefore, to make a proposal that only the weak type accepts, A sets

$$x^*_{7H} = \min\{1 + \delta\frac{1 - x^*_{3H}}{1-\delta} - \delta\left(\frac{1-p}{1-\delta}\right) + (1+\delta)b_H, 1\}.$$

To characterize A's best reply, we first determine the conditions under which A chooses to make a pooling over a separating proposal, conditional on the existence of both proposals. A does so when $U_A(x^*_{7L}) \geq U_A(x^*_{7H})$,

$$x^*_{7L} + \delta\frac{x^*_{4L}}{1-\delta} \geq \beta\left(-a + \delta\left(\frac{p}{1-\delta} - a\right)\right) + (1-\beta)\left(x^*_{7H} + \delta\frac{x^*_{3H}}{1-\delta}\right) \Leftrightarrow \beta \geq \frac{(1+\delta)(b_H - b_L)}{1 + (1+\delta)(a + b_H)} \equiv \hat{\beta}.$$

This threshold, $\hat{\beta}$, is guaranteed to fall in between $\underline{\beta}$ and $\overline{\beta}$, which define the case, as long as $b_H > b_L$, which is true by assumption.

Next, we determine the conditions under which each proposal exists in Case 5. We begin with the pooling proposal, and as before, we compare it to the case in which B promises as much as it can in the current period, $x_{7L} = 1$, such that A prefers attack to *any* pooling proposal iff

$$-a + \delta\left[\beta\left(\frac{p}{1-\delta} - a\right) + (1-\beta)\left(x^*_{6H} + \delta\frac{x^*_{1H}}{1-\delta}\right)\right] > 1 + \delta\frac{x^*_{4L}}{1-\delta},$$

or when

$$s > -\frac{(-1+\delta)\left(1 + a + \delta - p\delta + a\beta\delta + (-1+\beta)\delta b_H + \delta(1+\delta)b_L\right)}{-1 + \delta(p+\delta) + a(-1+\delta)(1+\beta\delta) + (-1+\delta)\delta\left((-1+\beta)b_H + (1+\delta)b_L\right)}.$$

Algebra again shows that the attack constraint above falls above $\hat{s}$ as long as $0 < \delta < 1$, which is true by construction. Therefore, there always exists a pooling proposal that A prefers to attack in Case 5.

We next compare attack to a separating proposal in which B promises as much as it can in the current period, $x_{7H} = 1$, such that A prefers attack to *any* separating proposal iff

$$-a + \delta\left[\beta\left(\frac{p}{1-\delta} - a\right) + (1-\beta)\left(x^*_{6H} + \delta\frac{x^*_{1H}}{1-\delta}\right)\right] > \beta\left[-a + \delta\left(\frac{p}{1-\delta} - a\right)\right] + (1-\beta)\left(1 + \delta\frac{x^*_{3H}}{1-\delta}\right),$$

or when

$$s > -1 + \frac{p\delta^2}{-1 + a(-1+\delta) + \delta(p+\delta) + (-1+\delta)\delta^2 b_L}.$$

Algebra again shows that the attack constraint above falls above $\hat{s}$ as long as $0 < \delta < 1$, which is true by construction, ensuring that there always exists a separating proposal that A prefers to attack in Case 5. Therefore, in Case 5, A never attacks. When $\beta \geq \hat{\beta}$, A proposes x^*_{7L} and both types of B accept, and when $\beta < \hat{\beta}$, A proposes x^*_{7H}, which only the weak type accepts.

Case 6. $\underline{\beta} \leq \beta < \overline{\beta}$ **and** $s > \hat{s}$**.**

Post-battle, pre-shift: attack regardless of beliefs. *Post-shift*: if both types, make a separating after one battle but a pooling proposal after no battles; if one type, satisfy it.

First, define a pooling proposal as one that both types of B accept, which implies that any type's rejection is out of equilibrium. This engages A's out-of-equilibrium beliefs, $\beta' = 1$. The strong type accepts when

$$1 - x_{7L} + \delta\frac{1-x^*_{4L}}{1-\delta} \geq -b_L + \delta\left(\frac{1-p}{1-\delta} - b_L\right), \Leftrightarrow x_{7L} \leq 1 + \delta\frac{1-x^*_{4L}}{1-\delta} - \delta\left(\frac{1-p}{1-\delta}\right) + (1+\delta)b_L.$$

The weak type accepts when

$$1 - x_{7L} + \delta\frac{1-x^*_{4L}}{1-\delta} \geq -b_H + \delta\left(\frac{1-p}{1-\delta} - b_H\right).$$

As before, the weak type's payoffs differ only in its lower payoff to rejection, which ensures that it is sure to accept all proposals also acceptable to the strong type. To make a pooling proposal that both types accept, A sets

$$x^*_{7L} = \min\{1 + \delta\frac{1-x^*_{4L}}{1-\delta} - \delta\left(\frac{1-p}{1-\delta}\right) + (1+\delta)b_L, 1\}.$$

Next, define a separating proposal as one that only the weak type of B accepts, leading A to believe $\beta = 1$ upon rejection and $\beta = 0$ upon acceptance. The incentive compatibility constraints are

$$1 - x_{7H} + \delta \frac{1 - x_{3H}^*}{1-\delta} \geq -b_H + \delta \left(\frac{1-p}{1-\delta} - b_H \right) \tag{b_H}$$

$$-b_L + \delta \left(\frac{1-p}{1-\delta} - b_L \right) > 1 - x_{7H} + \delta(-b_L) + \delta^2 \frac{1 - x_{1L}^*}{1-\delta} \tag{b_L}$$

which reduce to

$$1 + \delta^2 \frac{1 - x_{1L}^*}{1-\delta} + b_L - \delta \left(\frac{1-p}{1-\delta} \right) < x_{7H} \leq 1 + \delta \frac{1 - x_{3H}^*}{1-\delta} - \delta \left(\frac{1-p}{1-\delta} \right) + (1+\delta) b_H.$$

This inequality holds, ensuring the existence of a separating proposal, when $b_L < b_H$, which is true by construction. Therefore, to make a proposal that only the weak type accepts, A sets

$$x_{7H} = \min\{1 + \delta \frac{1 - x_{3H}^*}{1-\delta} - \delta \left(\frac{1-p}{1-\delta} \right) + (1+\delta) b_H, 1\}.$$

We begin the characterization of A's best response by noting that the same threshold divides its willingness to make separating and pooling proposals that divides it in Case 5, or $\hat{\beta}$. (Indeed, each side's payoffs for each action and the relevant proposals are also identical across cases.) Therefore, to complete A's strategy in Case 5, we again determine the conditions under which each proposal exists as a function of s.

Begin with the pooling proposal. The most that B can promise in the current period is $x_{7L} = 1$, which implies that A prefers attack to any pooling proposal when

$$-a + \delta \left(\frac{p}{1-\delta} - a \right) > 1 + \delta \frac{x_{4L}^*}{1-\delta} \Leftrightarrow s > \tilde{s},$$

as defined in Case 3. Otherwise, when $s \leq \tilde{s}$, a pooling proposal that A prefers to make is sure to exist. Now consider the separating proposal. The most that B can promise in the current period is $x_{7H} = 1$, which implies that A refers attack to any separating proposal when

$$-a + \delta \left(\frac{p}{1-\delta} - a \right) > \beta \left[-a + \delta \left(\frac{p}{1-\delta} - a \right) \right] + (1-\beta) \left(1 + \delta \frac{x_{3H}^*}{1-\delta} \right) \Leftrightarrow s > \bar{s},$$

as defined in Case 3. Otherwise, a separating proposal that A prefers to make is sure to exist.

Therefore, the following are A's best responses in Case 6. When $\beta \geq \hat{\beta}$ and $s \leq \tilde{s}$, both proposals exist, and A makes the pooling x_{7L}^*, which both types of B accept. When $\beta < \hat{\beta}$ and $s \leq \tilde{s}$, both proposals exist, and A makes the separating x^*_{7H}, which only the strong B rejects. When $\tilde{s} < s \leq \overline{s}$, only the pooling proposal exists, which A prefers to attack and therefore makes. Finally, when $s > \overline{s}$, there exists no proposal that A prefers to attack, and it attacks. A only updates its beliefs after proposing x^*_{7H}.

Case 7. $\beta < \underline{\beta}$ **and** $s \leq \underline{s}$**.**

Post-battle, pre-shift: if both types, make a separating proposal; if one type, satisfy it. *Post-shift*: if both types, make a separating proposal; if one type, satisfy it.

First, define a pooling proposal as one that both types of B accept, which implies that rejection is out of equilibrium. This engages A's out-of-equilibrium beliefs, $\beta' = 1$. The strong type accepts when

$$1 - x_{7L} + \delta(-b_L) + \delta^2\left(\frac{1 - x^*_{1L}}{1-\delta}\right) \geq -b_L + \delta(1 - x^*_{5L}) + \delta^2\frac{1 - x^*_{1L}}{1-\delta} \Leftrightarrow x_{7L} \leq 1 - \delta\,(b_L) + b_L - \delta(1 - x^*_{5L}).$$

The weak type accepts when

$$1 - x_{7L} + \delta(1 - x^*_{4H}) + \delta^2\frac{1 - x^*_{3H}}{1-\delta} \geq -b_H + \delta(1 - x^*_{5L}) + \delta^2\frac{1 - x^*_{1L}}{1-\delta},$$

or when

$$x_{7L} \leq 1 + \delta(1 - x^*_{4H}) + \delta^2\frac{1 - x^*_{3H}}{1-\delta} + b_H - \delta(1 - x^*_{5L}) - \delta^2\frac{1 - x^*_{1L}}{1-\delta}.$$

The weak type's acceptance constraint falls above the strong's as long as $b_H > b_L$, which ensures that it is sure to accept all proposals also acceptable to the strong type. To make a pooling proposal that both types accept, A sets

$$x^*_{7L} = \min\{1 - \delta\,(b_L) + b_L - \delta(1 - x^*_{5L}), 1\}.$$

Next, define a separating proposal as one that only the weak type of B accepts, leading A to believe $\beta = 1$ upon rejection and $\beta = 0$ upon acceptance.The incentive compatibility constraints are

$$1 - x_{7H} + \delta \frac{1 - y^*_{h,3}}{1-\delta} \geq -b_H + \delta(1 - x^*_{5L}) + \delta^2 \frac{1 - x^*_{1L}}{1-\delta} \qquad (b_H)$$

$$-b_L + \delta(1 - x^*_{5L}) + \delta^2 \frac{1 - x^*_{1L}}{1-\delta} > 1 - x_{7H} + \delta(-b_L) + \delta^2 \frac{1 - x^*_{1L}}{1-\delta} \qquad (b_L),$$

which reduce to

$$1 + \delta(-b_L) + b_L - \delta(1 - x^*_{5L}) < x_{7H} \leq 1 + \delta \frac{1 - x^*_{3H}}{1-\delta} - \delta(1 - x^*_{5L}) - \delta^2 \frac{1 - x^*_{1L}}{1-\delta} + b_H.$$

This inequality holds, ensuring the existence of a separating proposal, when $b_H > b_L$, which is true by construction. Therefore, to make a separating proposal, A sets

$$x^*_{7H} = \min\{1 + \delta \frac{1 - x^*_{3H}}{1-\delta} - \delta(1 - x^*_{5L}) - \delta^2 \frac{1 - x^*_{1L}}{1-\delta} + b_H, 1\}.$$

To characterize A's best reply, we first show that A never chooses to make a pooling over a separating proposal when both exist. A would make the pooling proposal iff $U_A(x^*_{7L}) \geq U_A(x^*_{7H})$, where

$$U_A(x^*_{7L}) = x^*_{7L} + \delta \left[\beta \left(-a + \delta \frac{x^*_{1L}}{1-\delta} \right) + (1-\beta) \left(x^*_{4,h} + \delta \frac{x^*_{3H}}{1-\delta} \right) \right]$$

and

$$U_A(x^*_{7H}) = \beta \left(-a + \delta x^*_{5L} + \delta^2 \frac{x^*_{1H}}{1-\delta} \right) + (1-\beta) \left(x^*_{7H} + \delta \frac{x^*_{3H}}{1-\delta} \right),$$

or iff $\beta \geq (b_H - b_L)/(1 + a + b_H) = \underline{\beta}$, which cannot hold since $\beta < \underline{\beta}$ defines the case. Therefore, the separating proposal dominates the pooling proposal in Case 7 as long as both exist.

To show that A only makes the separating proposal in Case 7, it is only necessary to show that it exists for the whole case, which guarantees that it makes the separating proposal whether the pooling exists or not. The most generous separating proposal that B can promise to A in the current period is $x_{7H} = 1$, and when $U_A(\text{att}_7) > U_A(x^*_{7H})$, there exists no separating proposal that A prefers to attacking. Therefore, A attacks iff

$$-a + \delta \left[\beta \left(\frac{p}{1-\delta} \right) + (1-\beta) \left(x^*_{6H} + \frac{x^*_{1H}}{1-\delta} \right) \right] > \beta \left(-a + \delta x^*_{5L} + \delta^2 \frac{x^*_{1L}}{1-\delta} \right) + (1-\beta) \left(1 + \delta \frac{x^*_{3H}}{1-\delta} \right)$$

or when

$$s > \frac{(-1+\delta)\left(a(1+\beta(-1+\delta))+(-1+\beta)(-1+(-1+p)\delta)+\delta(\beta+\delta-\beta\delta)b_L\right)}{-a(1+\beta(-1+\delta))(-1+\delta)+(-1+\beta)(-1+\delta(p+\delta))+(\beta(-1+\delta)-\delta)(-1+\delta)\delta b_L}.$$

Algebra shows that this constraint is greater than $\underline{s}$ (the constraint that defines the case) as long as $0 < \delta < 1$, which is true by construction. Therefore, there always exists a separating proposal in Case 7 that A prefers to attack. A best response in Case 7 is thus to propose x^*_{7H}, which the strong B rejects and the weak accepts, allowing A to update its beliefs.

Case 8. $\beta < \underline{\beta}$ **and** $\underline{s} < s \leq \hat{s}$**.**

Post-battle, pre-shift: if both types, make a separating proposal; if one type, attack the strong but satisfy the weak. *Post-shift*: if both types, make a separating proposal; if one type, satisfy it.

First, define a pooling proposal as one that both types of B accept, which implies that rejection is out of equilibrium. This engages A's out-of-equilibrium beliefs, $\beta' = 1$. The strong type accepts when

$$1 - x_{7L} + \delta(-b_L) + \delta^2 \frac{1-x^*_{1L}}{1-\delta} \geq -b_L + \delta\left(\frac{1-p}{1-\delta} - b_L\right) \Leftrightarrow x_{7L} \leq 1 + b_L + \delta^2 \frac{1-x^*_{1L}}{1-\delta} - \delta\left(\frac{1-p}{1-\delta}\right).$$

The weak type accepts when

$$1 - x_{7L} + \delta(1-x^*_{4H}) + \delta^2 \frac{1-x^*_{3H}}{1-\delta} \geq -b_H + \delta\left(\frac{1-p}{1-\delta} - b_H\right),$$

or when

$$x_{7H} \leq 1 + \delta(1-x^*_{4H}) + \delta^2 \frac{1-x^*_{3H}}{1-\delta} + b_H - \delta\left(\frac{1-p}{1-\delta} - b_H\right).$$

The weak type's acceptance constraint falls above the strong's as long as $b_H > b_L$, which ensures that it is sure to accept all proposals also acceptable to the strong type. To make a pooling proposal that both types accept, A sets

$$x^*_{7L} = \min\{1 + b_L + \delta^2 \frac{1-x^*_{1L}}{1-\delta} - \delta\left(\frac{1-p}{1-\delta}\right), 1\}.$$

Next, define a separating proposal as one that only the weak type of B accepts, leading A to believe $\beta = 1$ upon rejection and $\beta = 0$ upon acceptance. The incentive compatibility constraints are

$$1 - x_{7H} + \delta \frac{1 - x^*_{3H}}{1-\delta} \geq -b_H + \delta \left(\frac{1-p}{1-\delta} - b_H \right) \qquad (b_H)$$
$$-b_L + \delta \left(\frac{1-p}{1-\delta} - b_L \right) > 1 - x_{7H} + \delta(-b_L) + \delta^2 \frac{1 - x^*_{1L}}{1-\delta} \qquad (b_L),$$

which reduce to

$$1 + \delta^2 \frac{1 - x^*_{1L}}{1-\delta} + b_L - \delta \left(\frac{1-p}{1-\delta} \right) < x_{7H} \leq 1 + \delta \frac{1 - x^*_{3H}}{1-\delta} - \delta \left(\frac{1-p}{1-\delta} \right) + (1+\delta) b_H.$$

This inequality holds, ensuring the existence of a separating proposal, when $b_H > b_L$, which is true by construction. Therefore, to make a separating proposal, A sets

$$x^*_{7H} = \min\{1 + \delta \frac{1 - x^*_{3H}}{1-\delta} - \delta \left(\frac{1-p}{1-\delta} \right) + (1+\delta) b_H, 1\}.$$

To characterize A's best reply, we first state the conditions under which A chooses to make a separating over a pooling proposal when both proposals exist. A does so iff $U_A(x^*_{7,h}) > U_A(x^*_{7,l})$, where

$$U_A(x^*_{7H}) = \beta \left(-a + \delta x^*_{5L} + \delta^2 \frac{x^*_{1,l}}{1-\delta} \right) + (1-\beta) \left(x^*_{7H} + \delta \frac{x^*_{3H}}{1-\delta} \right)$$

and

$$U_A(x^*_{7L}) = x^*_{7L} + \delta \left(\beta \left(-a + \delta \frac{x^*_{1L}}{1-\delta} \right) + (1-\beta) \left(x^*_{4H} + \delta \frac{x^*_{3H}}{1-\delta} \right) \right),$$

or when $\beta < (b_H - b_L)/(1 + a - \delta + b_H)$. This constraint is strictly greater than $\underline{\beta}$, which defines the case. Therefore, A always prefers a separating to a pooling proposal in Case 8 when both exist.

To show that A makes the separating proposal in Case 8, it is only necessary to show that it exists for the whole case, which guarantees that it makes the separating proposal whether the pooling exists or not. The most that B can promise in a separating proposal is $x_{7H} = 1$, and when A prefers attacking even to a maximally generous proposal, then there exists no separating proposal that can prevent it from attacking. Therefore, A attacks iff

$$-a+\delta\left(\beta\left(\frac{p}{1-\delta}-a\right)+(1-\beta)\left(x_{6H}^{*}+\delta\frac{x_{1H}^{*}}{1-\delta}\right)\right)>U_A(x_{7H}^{*}),$$

or when

$$s>-1+\frac{p\delta^2}{-1+a(-1+\delta)+\delta(p+\delta)+(-1+\delta)\delta^2 b_L}.$$

Algebra shows that this constraint falls above $\hat{s}$, which defines the case, as long as $0<\delta<1$, guaranteeing that in Case 8 there always exists a separating proposal that A prefers to attacking. Therefore, in Case 8, A proposes x_{7H}^{*}, which the strong B rejects and the weak accepts, allowing A to update its beliefs.

Case 9. $\beta<\underline{\beta}$ **and** $s>\hat{s}$**.**

Post-battle, pre-shift: attack regardless of beliefs. *Post-shift*: if both types, make a separating proposal; if one type, satisfy it.

First, define a pooling proposal as one that both types of B accept, which implies that rejection is out of equilibrium. This engages A's out-of-equilibrium beliefs, $\beta'=1$. The strong type accepts iff

$$1-x_{7L}+\delta(-b_L)+\delta^2\frac{1-x_{1L}^{*}}{1-\delta}\geq -b_L+\delta\left(\frac{1-p}{1-\delta}-b_L\right),\Leftrightarrow x_{7L}\leq 1+b_L+\delta^2\frac{1-x_{1L}^{*}}{1-\delta}-\delta\left(\frac{1-p}{1-\delta}\right).$$

The weak type also accepts iff

$$1-x_{7L}+\delta(1-x_{4H}^{*})+\delta^2\frac{1-x_{3H}^{*}}{1-\delta}\geq -b_H+\delta\left(\frac{1-p}{1-\delta}-b_H\right),$$

or when

$$x_{7L}\leq 1+\delta(1-x_{4H}^{*})+\delta^2\frac{1-x_{3H}^{*}}{1-\delta}+b_H-\delta\left(\frac{1-p}{1-\delta}-b_H\right).$$

The weak type's acceptance constraint falls above the strong's as long as $b_H > b_L$, which ensures that it is sure to accept all proposals also acceptable to the strong type. To make a pooling proposal that both types accept, A sets

$$x_{7L}^* = \min\{1 + b_L + \delta^2 \frac{1 - x_{1L}^*}{1-\delta} - \delta\left(\frac{1-p}{1-\delta}\right), 1\}.$$

Next, define a separating proposal as one that only the weak type of B accepts, leading A to believe $\beta = 1$ upon rejection and $\beta = 0$ upon acceptance. The incentive compatibility constraints are

$$1 - x_{7H} + \delta\frac{1 - x_{3H}^*}{1-\delta} \geq -b_H + \delta\left(\frac{1-p}{1-\delta} - b_H\right) \qquad (b_H)$$
$$-b_L + \delta\left(\frac{1-p}{1-\delta} - b_L\right) > 1 - x_{7H} + \delta(-b_L) + \delta^2\frac{1 - x_{1L}^*}{1-\delta} \qquad (b_L),$$

which reduce to

$$1 + \delta^2\frac{1 - x_{1L}^*}{1-\delta} + b_L - \delta\left(\frac{1-p}{1-\delta}\right) < x_{7H} \leq 1 + (1+\delta)b_H + \delta\frac{1 - x_{3H}^*}{1-\delta} - \delta\left(\frac{1-p}{1-\delta}\right).$$

This inequality holds, ensuring the existence of a separating proposal, when $b_H > b_L$, which is true by construction. Therefore, to make a separating proposal, A sets

$$x_{7H}^* = \min\{1 + (1+\delta)b_H + \delta\frac{1 - x_{3H}^*}{1-\delta} - \delta\left(\frac{1-p}{1-\delta}\right), 1\}.$$

To characterize A's best reply, we first demonstrate that it always prefers to make a separating over a pooling proposal in this case when both proposals exist. A prefers the separating proposal iff $U_A(x_{7H}^*) > U_A(x_{7L}^*)$, where

$$U_A(x_{7H}^*) = \beta\left[-a + \delta\left(\frac{p}{1-\delta} - a\right)\right] + (1-\beta)\left(x_{7H}^* + \delta\frac{x_{3H}^*}{1-\delta}\right)$$

and

$$U_A(x_{7L}^*) = x_{7L}^* + \delta\left[\beta\left(-a + \delta\frac{x_{1L}^*}{1-\delta}\right) + (1-\beta)\left(x_{4H}^* + \delta\frac{x_{3H}^*}{1-\delta}\right)\right],$$

or when $\beta < (b_H - b_L)/(1 + a - \delta + b_H)$, which is strictly greater than $\underline{\beta}$, which defines the case. So A always prefers separating to pooling proposals in Case 9 when both exist. As long as the separating proposal exists, A will always make it, regardless of the existence of the pooling proposal. Finally, we need only determine when such a separating proposal exists. The most that B can promise in such a proposal is $x_{7H} = 1$, and when A prefers attack to it, there exists no such deal that can prevent attack. Therefore, A attacks iff

$$-a+\delta\left(\frac{p}{1-\delta}-a\right) > \beta\left[-a+\delta\left(\frac{p}{1-\delta}-a\right)\right]+(1-\beta)\left(1+\delta\frac{x_{3H}^*}{1-\delta}\right),$$

or when $s > \overline{s}$, as defined in Cases 3 and 6. Therefore, in Case 9, A proposes x_{7H}^*, which the strong type rejects and the weak accepts, when $s \leq \underline{s}$, and it attacks when $s > \overline{s}$. A updates its beliefs only after making the separating proposal.

Comparative Statics

Proof of Result 4. This is confirmed by

$$\frac{\partial \overline{s}}{\partial b_H} = \frac{p(1-\delta)\delta^3}{\left(\delta(p+\delta)-a\left(1-\delta^2\right)-(1-\delta)\delta\left(b_H+\delta b_L\right)-1\right)^2} > 0$$

and

$$\frac{\partial \underline{s}}{\partial b_L} = \frac{p(1-\delta)\delta^2}{\left(p-a(1-\delta)+\delta-(1-\delta)\delta b_L-1\right)^2} > 0.$$

Therefore, as b_H increases or b_L decreases, the difference $\overline{s}-\underline{s}$ increases. □

Proof of Result 5. To see this, note that $\partial\overline{s}/\partial a > \partial\underline{s}/\partial a > 0$, or

$$\frac{p\delta^2\left(1-\delta^2\right)}{\left(\delta(p+\delta)-a\left(1-\delta^2\right)-(1-\delta)\delta\left(b_H+\delta b_L\right)-1\right)^2} > \frac{p(1-\delta)\delta}{\left(p-a(1-\delta)+\delta-(1-\delta)\delta b_L-1\right)^2} > 0.$$

□

Proof of Result 6. To see this, note that $\partial\overline{s}/\partial p < \partial\underline{s}/\partial p < 0$, or

$$\frac{-(1-\delta)\delta^2\left(1+a+\delta+a\delta+\delta b_H+\delta^2 b_L\right)}{\left(\delta(p+\delta)-a\left(1-\delta^2\right)-(1-\delta)\delta\left(b_H+\delta b_L\right)-1\right)^2} < \frac{-(1-\delta)\delta\left(1+a+\delta b_L\right)}{\left(p-a(1-\delta)+\delta-(1-\delta)\delta b_L-1\right)^2} < 0.$$

□